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Elementary Proof of Zeeman's Theorem

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We prove that the nonaffine conformal transformations of four-dimensional Minkowski space are necessarily global causality violators, and use this fact to obtain an elementary group-theoretic proof of the theorem that global causality implies the Lorentz group.

1. INTRODUCTION

The proof of the theorem that causality implies the Lorentz group was first given by Zeeman (1964). In a previous paper (Briginshaw, 1980), the author has explained how the class of causal automorphisms is contained in the conformal group and pointed out how the presence of the inversion map among the generators of the conformal group shows itself by its violation of global causality. It seems natural, therefore, to use this fact to fabricate an elementary group-theoretic proof of the causality theorem; that is exactly the concern of the present paper.

Notation. Let M_4 denote the vector space of quadruples of real numbers of the type (x_0, x_1, x_2, x_3) with the (indefinite) inner product . given by

$$x \cdot y = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3$$

We define two relations in M_4, as does Zeeman (1964): (i) $x < y$ iff $(x-y)\cdot(x-y) > 0$ and $x_0 < y_0$ (this is a partial order); (ii) $x <\cdot y$ iff $(x-y)\cdot(x-y) = 0$ and $x_0 < y_0$ (this is not a partial order). We denote the pair $(M_4, <)$ by $\hat{M}_4$ and the pair $(M_4, <\cdot)$ by $\dot{M}_4$.

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Causal Automorphisms

A map $f:\hat{M}_4 \rightarrow \hat{M}_4$ is said to be an *automorphism* of $\hat{M}_4$ if f is a bijection of $\hat{M}_4$ such that both f, f^{-1} preserve $<$, i.e., $x<y \Leftrightarrow fx<fy$.

A map $f: \hat{\dot{M}}_4 \rightarrow \hat{\dot{M}}_4$ is similarly an *automorphism* of $\hat{\dot{M}}_4$ if f is bijective and such that $x<\cdot y \Leftrightarrow fx<\cdot fy$.

Each collection of automorphisms is a group under composition, and we shall refer to the group of automorphisms of $\hat{M}_4$ as G and the group of automorphisms of $\hat{\dot{M}}_4$ as $\dot{G}$.

Lemma 1

$$G=\dot{G}$$

This is the same result as Lemma 1 of Zeeman (1964).

2. THE INTERVAL TOPOLOGY

Define $0_{x,y}=\{z: x<z<y\}$; then the collection $\{\{0_{x,y}\}: x, y \in M_4\}$ is a base of neighborhoods for a topology on M_4, that we call the *interval topology* (Nanda, 1976).

We shall denote the interval topology by I and the Euclidean metric topology by E; we now have the following two results as stated and proved by Nanda (1976).

Lemma 2

$$I=E$$

This is Lemma 1 of Nanda (1976).

Lemma 3. If $f \in G$ then f is a homeomorphism of (M_4, E).

This is Lemma 2 of Nanda (1976).

Corollary. If $f \in \dot{G}$ then f is a homeomorphism of (M_4, E).

Proof.

$$f \in \dot{G} \Rightarrow f \in G.$$

Definition: Null Line. The set $N_{p,q}=\{x: x=p+\mu b, \mu \in R, p-q=b, b \cdot b=0\}$ is said to be the *null line* through p and q.

Lemma 4. There are no null line triangles in M_4.

Proof. Without loss of generality we may prove that if $0 <\cdot \alpha <\cdot \beta$, and if $0 <\cdot \beta$ then $0, \alpha, \beta$ are points of the same null line. Write $\alpha - \beta = p$, so that $p \cdot p = 0$; now $\beta + p$ is a null vector, therefore $\beta \cdot \beta + 2\beta \cdot p + p \cdot p = 0$, i.e., $\beta \cdot p = 0$. If β, p are not in the same direction, then

$$|\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3| < (\beta_1^2 + \beta_2^2 + \beta_3^2)^{1/2} \cdot (p_1^2 + p_2^2 + p_3^2)^{1/2} = |\beta_0||p_0|$$
$$= \beta_0 p_0$$
$$\therefore \beta \cdot p > 0$$

Thus $0, \alpha, \beta$ are collinear.

Lemma 5. If $f \in \dot{G}$ then f maps every null line onto a null line.

Proof. Let N_p be an arbitrary null line through p, and let $q \in N_p$ be such that $p <\cdot q$. Suppose that p, q are fixed and that $\xi \in N_p$ is such that $p <\cdot \xi <\cdot q$. Since $f \in \dot{G}$ it follows that $fp <\cdot f\xi <\cdot fq$, and that $fp, f\xi, fq$ are collinear.

Definition: Conformal Map. Let f: $\overline{M}_4 \to \overline{M}_4$ (compactified M_4) be such that f is continuous at every nonsingular point and such that for all appropriate x, y and their maps $x' = fx$, $y' = fy$, we have

$$\frac{dx' \cdot dy'}{(dx' \cdot dx')^{1/2}(dy' \cdot dy')^{1/2}} = \frac{dx \cdot dy}{(dx \cdot dx)^{1/2}(dy \cdot dy)^{1/2}}$$

Then f is said to be a *conformal map* of $\overline{M}_4$; under composition the collection of such maps is a group, called the *conformal group*, C.

Definition: The Inversion Map. The conformal map I: $\overline{M}_4 \to \overline{M}_4$ given by $Ix = x/(x \cdot x)$ is said to be an *inversion* of $\overline{M}_4$.

3. GENERATORS OF THE CONFORMAL GROUP

In the first of a series of papers concerning the covariance group associated with Maxwell's equations, Bateman (1909) proved that the conformal group is that generated by dilatations, Poincaré (inhomogeneous Lorentz) transformations, and inversions. In the next lemma we prove that I is a global causality violator.

Lemma 6

$$I \notin \dot{G}$$

Proof. Consider the null line $N=\{x\colon x=p+\mu b, \mu\in R, b\cdot b=0, b\neq 0\}$. Now, if $x\in N$, then

$$Ix=\frac{p+\mu b}{p\cdot p+2\mu b}=\frac{p}{p\cdot p}+\frac{\mu[(p\cdot p)b-2(p\cdot b)p]}{(p\cdot p)[p\cdot p+2\mu(p\cdot b)]}$$

and there is clearly one singular point on N, where the value of μ is

$$\mu^*=\frac{-p\cdot p}{2p\cdot b}$$

In addition, the expression for Ix has the form

$$Ix=\frac{p}{p\cdot p}+\alpha X$$

where $X=(p\cdot p)b-2(p\cdot b)p$ is null. Thus the map of a null line is a null line, as expected; in addition

$$\alpha=\frac{1}{2(p\cdot p)(p\cdot b)}\left[1-\frac{p\cdot p}{p\cdot p+2\mu(p\cdot b)}\right]$$

Now p is an arbitrary point on N and we may choose it such that $0<p$, i.e., $p\cdot p>0$ and $p_0>0$; equally well, without loss of generality, we may consider $0<\cdot b$, i.e., $b\cdot b=0$ and $b_0>0$, in which case $p\cdot b>0$. Now, if $\mu_1<\mu_2<\mu^*$, or if $\mu^*<\mu_1<\mu_2$, then $\alpha_1<\alpha_2$, and the relation $<\cdot$ is preserved. On the other hand, if $\mu_1<\mu^*<\mu_2$, then $\alpha_2<\alpha_1$, and the relation $<\cdot$ is violated.

Notation. We refer to the affine group generated by translations, orthochronous Lorentz transformations and dilatations as $P\uparrow^*$.

Remark. It is easy to prove the following catalog of results concerning the generators of C: if λ, t_a, Λ are, respectively, arbitrary dilatations, translations, and Lorentz transformations, then

(i) $\lambda\circ I=I\circ\lambda^{-1}$
(ii) $\Lambda\circ I=I\circ\Lambda$
(iii) $\lambda\circ\Lambda=\Lambda\circ\lambda$
(iv) $\Lambda\circ t_a=t_{\Lambda a}\circ\Lambda$
(v) $\lambda\circ t_a=t_{\lambda a}\circ\lambda$

There is no similar commutation relation for t_a and I. We now prove the main theorem.

Theorem

$$p\uparrow^*=\dot{G}$$

Proof. It is clear that $P\uparrow^* \subseteq \dot{G}$. Also, if $f \in \dot{G}$, then f is continuous, from Lemma 3; and f preserves null lines, from Lemma 5; from Weyl (1923) it follows that $f \in C$. Now, if $f = \alpha_1 \circ \alpha_2 \circ \alpha_3 \circ \alpha_4 \circ \cdots \circ \alpha_N$, where each α_i is an element of the generating subgroups of dilatation, translation, Lorentz, and inversion maps, then we may arrange f thus:

$$f = \lambda \circ \Lambda \circ t_a \circ I \circ t_b \circ I \circ t_c \circ I \circ \cdots \circ t_p \circ I \circ t_q$$

where λ, Λ are, respectively, a dilatation and a Lorentz transformation and $t_a, t_b, \ldots, t_q$ are translations. Consider the image of an arbitrary null line under f; without loss of generality we may take $q = 0$, and consider the line $N = \{x: x = \xi + \mu\eta,\ \text{where}\ \xi \cdot \xi \neq 0, \eta \cdot \eta = 0, \eta \neq 0\}$. Suppose at least one $b, \ldots, p$ is zero; we write $K(y) = \{x:(x-y)\cdot(x-y) = 0\}$ and refer to it as the *null cone* at y. It is clear that, for each $f \in C$, with $b, \ldots, p$ nonzero, there is a corresponding network of finitely many singular surfaces. Thus, I is singular on $K(0)$ (as is $t_p \circ I$), $I \circ t_p \circ I$ is singular on $K(0) \cup K(-p/p\cdot p)$ (as is $t_j \circ I \circ t_p \circ I$); if we now write $\Delta_{p,j} = 1 + 2(p\cdot j) + (p\cdot p)(j\cdot j)$, then $I \circ t_j \circ I \circ t_p \circ I$ is singular on $K(0) \cup K(-p/p\cdot p) \cup K(\{-p(j\cdot j)/\Delta_{p,j}\} - \{j/\Delta_{p,j}\})$, and so on.

Now consider the point x^* at which N intersects the cone $K(0)$, and let μ^* be the parameter associated with x^* as in Lemma 6. Let μ_1, μ_2, μ_3 be close enough to μ^* to ensure that $x_1 <\cdot\, x^* <\cdot\, x_2 <\cdot\, x_3$ but that if S is the union of all singular surfaces for f, $\forall u, v \in S - K(0), u <\cdot\, x_1 <\cdot\, x_3 <\cdot\, v$.

Thus $Ix_2 <\cdot\, Ix_3 < Ix_1$; also, for every other map of the collection comprising the *nonaffine* part of f these relations are preserved.

Case (i): Suppose at least one of $b, c, \ldots, p \neq 0$ and that Λ is nonorthochronous, then $<\cdot$ is violated for the pair x_2, x_3.

Case (ii): Suppose at least one of $b, \ldots, p \neq 0$ and that Λ is orthochronous, then $<\cdot$ is violated for the pair x_1, x_2.

Case (iii): Suppose that every $b, \ldots, p = 0$ and that Λ is nonorthochronous, then $<\cdot$ is violated for every pair on N. We conclude that every $b, c, \ldots, p = 0$ and that Λ is orthochronous, in which case $f \in P\uparrow^*$.

Remark. For $n \geqslant 3$, the causal automorphisms of M_n are *always* elements of the conformal group. However, for $n = 2$, that is not true; there are *global* causal automorphisms of M_2 which are not conformal maps. Thus the theorem is not valid for $n = 2$.

REFERENCES

Bateman, H (1909). *Proceedings of the London Mathematical Society*, **7**, 70.
Briginshaw, A. J. (1980). *International Journal of Theoretical Physics*, **19**, 329.
Nanda, S. (1976). *Mathematical Proceedings of the Cambridge Philosophical Society*, **79**, 533.
Weyl, H. (1923). *Mathematicsche Analyse des Raumproblems*, Springer, Berlin, pp. 71–75.
Zeeman, E. C. (1964). *Journal of Mathematical Physics*, **5**, 490.